Equivalence of formulas and formal proofs

Definition.

Let $A[v_1, \ldots, v_n]$ be a formula of a language L. We say that A is universally valid, written $\models A$ if for any L-structure $\mathcal{M}$, $\mathcal{M} \models A$.

Also, two formulas A, B are logically equivalent written $A \equiv B$ if $(A \leftrightarrow B)$ is universally valid.

And B is a logical consequence of A if $\models A \to B$. We write $A \Rightarrow B$ for this.

Note If A and B have free variables, these become important. For example, the formulas

$$A : \ (Px \wedge Py) \longrightarrow Sxy$$

$$B : \ (Pz \wedge Pv) \longrightarrow Szv$$

will be true in the same structures. For

$$\mathcal{M} \models A \iff \mathcal{M} \models \forall x \forall y \left((Px \wedge Py) \to Sxy \right)$$

$$\iff \mathcal{M} \models \forall z \forall v \left((Pz \wedge Pv) \to Szv \right)$$

$$\iff \mathcal{M} \models B.$$

On the other hand, $A \not\equiv B$, since

$$\not\models \ ((Px \wedge Py) \rightarrow Sxy) \leftrightarrow ((Pz \wedge Pv) \rightarrow Szv)$$

i.e.

$$\not\models \forall x \forall y \forall z \forall v \ ((Px \wedge Py) \rightarrow Sxy) \leftrightarrow ((Pz \wedge Pv) \rightarrow Szv)$$

__Definition__ A formula A is said to be in __prenex form__ if there is a quantifier free formula B, variables $v_1, \dots, v_n$ and quantifiers $Q_1, \dots, Q_n$ s.t.

$$A = Q_1 v_1 \, Q_2 v_2 \cdots Q_n v_n \, B$$

Moreover, the prenex form is __nice__ if the variables $v_1, v_2, \dots, v_n$ are all distinct.

__Theorem__ Every formula A is logically equivalent with one in nice prenex form.

The proof is by induction on the construction of A.

For the proof, the following equivalences

If A, B and F are formulas and x, y are variables, and x is <u>not</u> free in F, then

$$\exists x\,(A \vee B) \equiv \exists x A \vee \exists x B$$

$$\forall x\,(A \wedge B) \equiv \forall x A \wedge \forall x B$$

$$\exists x\,(A \rightarrow B) \equiv \forall x A \rightarrow \exists x B$$

$$\forall x \forall y\, A \equiv \forall y \forall x\, A$$

$$\exists x \exists y\, A \equiv \exists y \exists x$$

$$\neg \forall x\, A \equiv \exists x \neg A$$

$$\exists x\,(A \wedge B) \Longrightarrow \exists x A \wedge \exists x B$$

$$\forall x A \vee \forall x B \Longrightarrow \forall x\,(A \vee B)$$

$$\exists x \forall y\, A \Longrightarrow \forall y \exists x\, A$$

$$\forall x\,(A \vee F) \equiv \forall x A \vee F$$

$$\exists x\,(A \wedge F) \equiv \exists x A \wedge F$$

$$\forall x\,(A \rightarrow F) \equiv \exists x A \rightarrow F$$

A formal proof system for predicate logic.

Deduction rules

Modus ponens: From formulas F and $F \to G$, deduce G

Generalization: From a formula F, deduce $\forall x F$, where x is any variable.

A formula A is said to be a tautology of first order logic if there are formulas $B_1, \dots, B_n$, propositional variables $P_1, \dots, P_n$, and propositional formula F in variables $P_1, \dots, P_n$ st.

- F is a tautology
- A is obtained from F by substitution of B_i for P_i.

Logical axioms

- Any tautology of first order logic is an axiom
- Any formula $(\exists x A \longleftrightarrow \neg \forall x \neg A)$ is an axiom

- Any formula $(\forall x (A \to B) \to (A \to \forall x B))$, where x is not free in A, is an axiom.
- Any formula $\forall x A \to A[t/x]$, where t is a term such that no variable of t becomes bound in $A[t/x]$.

Note that the last restriction is important.

Eg, if t is the variable y, then if A is the formula $\exists y Rxy$, the formula

$$\forall x \exists y Rxy \to \exists y Ryy$$

is not valid.

Recall A theory is a set of sentences, i.e., closed formulas.

Definition Given a theory T, a proof from T is a finite sequence $(A_0, A_1, \ldots, A_n)$ such that each A_i is either

- a logical axiom, or
- obtained from $A_0, \ldots, A_{i-1}$ by the deduction rules, or
- an element of T.

In this case, we say that $(A_0, \ldots, A_n)$ is a <u>proof</u> of A_n from T.

If A is provable from T, we say that A is a <u>theorem</u> of T and write $T \vdash A$.